

Probabilistic cellular automata with general alphabets letting a Markov chain invariant.

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Abstract

This paper is devoted to probabilistic cellular automata (PCA) on \mathbb{N} , \mathbb{Z} or $\mathbb{Z}/n\mathbb{Z}$, depending of two neighbors, with a general alphabet E (finite or infinite, discrete or not). We study the following question: under which conditions does a PCA possess a Markov chain as invariant distribution? Previous results in the literature give some conditions on the transition matrix (for positive rate PCA) when the alphabet E is finite. Here we obtain conditions on the transition kernel of PCA with a general alphabet E . In particular, we show that the existence of an invariant Markov chain is equivalent to the existence of a solution to a cubic integral equation.

One of the difficulties to pass from a finite alphabet to a general alphabet comes from some problems of measurability, and a large part of this work is devoted to clarify these issues.

1 Introduction

CA and PCA with finite alphabet

Cellular automata (CA), as described by Hedlund [8], are discrete local dynamical systems on a space $E^{\mathbb{L}}$ where $E = \{0, \dots, \kappa\}$ is a finite alphabet, the set of states of cells, and \mathbb{L} is a discrete lattice. Formally, a cellular automaton A is a tuple (\mathbb{L}, E, N, f) where

- \mathbb{L} is a lattice, called set of cells. In this paper, \mathbb{L} is \mathbb{N} , \mathbb{Z} or $\mathbb{Z}/n\mathbb{Z}$.
- N is the neighborhood function: for $i \in \mathbb{L}$, $N(i) = (i + l : l \in L)$ where $L \subset \mathbb{L}$ is finite. Each neighborhood has cardinality $|N| = |L|$. In the paper, $N(i) = (i, i + 1)$ when the lattice is \mathbb{N} or \mathbb{Z} and $N(i) = (i, i + 1 \bmod n)$ when the lattice is $\mathbb{Z}/n\mathbb{Z}$.
- f is the local rule. It is a function $f : E^{|N|} \rightarrow E$.

The CA $\mathbf{A} = (\mathbb{L}, E, N, f)$ defines a global function $F : E^{\mathbb{L}} \mapsto E^{\mathbb{L}}$ on the set of configurations $E^{\mathbb{L}}$. For any configuration $S_0 = (S_0(i) : i \in \mathbb{L})$, the image $S_1 = F(S_0)$ of S_0 by F is defined by, for any $j \in \mathbb{L}$,

$$S_1(j) = f((S_0(i) : i \in N(j))).$$

In words, the state of all cells are updated simultaneously and the state $S_1(j)$ of the cell j at time 1 depends only of the states $(S_0(i) : i \in N(j))$ of its neighborhood at time 0. Hence, the dynamic is the following: starting from an initial configuration $S_{t_0} \in E^{\mathbb{L}}$ at time t_0 , the successive states of the system are $(S_t : t \geq t_0)$ where $S_{t+1} = F(S_t)$. The sequence of configurations

$$S = (S_t = (S_t(i) : i \in \mathbb{L}), t \geq t_0)$$

is called the space-time diagram of \mathbf{A} .

Notation. In the following, the state $S_t(i)$ of the cell i at time t will be denoted $S(i, t)$.

Probabilistic cellular automata (PCA) with finite alphabets are generalizations of CA in which the states $(S(i, t) : i \in \mathbb{L}, t \geq t_0)$ are random variables (r.v.) defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$, each of the r.v. $S(i, t)$ taking a.s. its value in E . Seen as random process, S is equipped with the σ -field generated by the cylinders. To define PCA, the local rule f is replaced by a transition matrix T (of size $E^{|N|} \times E$) which gives the distributions of the state of a cell at time $t + 1$ conditionally on those of its neighborhood at time t :

$$P(S(j, t + 1) = b \mid (S(i, t) = a_i : i \in N(j))) = T((a_i : i \in N(j)); b).$$

Conditionally on S_t , the states $(S(j, t + 1) : j \in \mathbb{L})$ are independent (see Eq (1)).

The transition matrix T is then an array of non negative numbers satisfying, for any $(a_1, \dots, a_{|N|}) \in E^{|N|}$, $\sum_{b \in E} T((a_1, \dots, a_{|N|}); b) = 1$.

Formally, a PCA \mathbf{A} with a finite alphabet E is an operator $\mathcal{F} : \mathcal{M}(E^{\mathbb{L}}) \mapsto \mathcal{M}(E^{\mathbb{L}})$ on the set of probability distributions $\mathcal{M}(E^{\mathbb{L}})$ on the set of configurations. If S_0 has distribution μ_0 , then S_1 has distribution $\mu_1 = \mathcal{F}(\mu_0)$. We can, also, define μ_1 directly from μ_0 and T , using Kolmogorov extension theorem (μ_1 is characterized by its finite-dimensional distributions), by: for any finite subset $C \subset \mathbb{L}$ and for any $(b_j : j \in C) \in E^C$,

$$\mu_1((b_j : j \in C)) = \sum_{(a_i)_{i \in N(C)} \in E^{N(C)}} \mu_0((a_i : i \in N(C))) \prod_{j \in C} T((a_i : i \in N(j)); b_j) \quad (1)$$

where $N(C) = \bigcup_{j \in C} N(j)$. A measure $\mu \in \mathcal{M}(E^{\mathbb{L}})$ is said to be *invariant* by \mathbf{A} if $\mathcal{F}(\mu) = \mu$.

The simplest case of PCA is the two colors case $E = \{0, 1\}$ on \mathbb{Z} with neighborhood $N(i) = (i, i + 1)$. They have been deeply studied and lots of results about them are known, see Toom [13]. For example, Belyaev [2] characterized the set of PCA possessing as invariant distribution a Markov chain indexed by \mathbb{Z} . Nevertheless, there are still interesting open

problems about them: for instance, the question whether all positive rate PCA (i.e., for any $a, b, c \in \{0, 1\}$, $T(a, b; c) > 0$) are ergodic or not is still open.

So far, it has been observed in different frameworks that explicit calculus of the invariant distribution of PCA can be done only if the transition matrix satisfies some algebraic equations (that forms a manifold in terms of the $(T(a, b; c) : a, b, c \in E)$). In Belyaev [2] this is shown for PCA with 2-letter alphabet whose invariant distributions are Markov chains or product measures. In Dai-Pra [12], this is done for PCA on \mathbb{Z}^d with a 2-letter alphabet and whose invariant distributions are Gibbs measures. And, in Casse and Marckert [5], the same phenomenon is observed for PCA on \mathbb{Z} or $\mathbb{Z}/n\mathbb{Z}$ with a finite alphabet letting a Markov chain invariant. Hence, literature focuses on characterizing PCA having simple invariant measures: product measures and Markov chains for $|N| = 2$ and Gibbs measures for PCA on \mathbb{Z}^d . In addition to [2], the study of PCA on \mathbb{Z} admitting an invariant product measure have been done by Mairesse and Marcovici [10] (in a finite alphabet case). For PCA letting a Markov chain invariant, in addition to [2] and [5], Bousquet-Mélou [4] characterizes those on $\mathbb{Z}/n\mathbb{Z}$ with a 2-letter alphabet and Toom [13] gives a sufficient condition for PCA on \mathbb{Z} with a finite alphabet.

The most general results are given in [5] where it is proved (in Theorem 2.6) that a positive PCA on \mathbb{Z} with two neighbors and a finite alphabet $E = \{0, \dots, \kappa\}$ admits an horizontal zigzag Markov chain (see Definition 1.6) as invariant distribution if and only if the two following conditions are satisfied:

1. for any $a, b, c \in E$,

$$T(a, b; c)T(a, 0; 0)T(0, b; 0)T(0, 0; c) = T(a, b; 0)T(a, 0; c)T(0, b; c)T(0, 0; 0) \text{ and}$$

2. $D^\gamma U^\gamma = U^\gamma D^\gamma$

$$\text{where } D^\gamma(a; c) = \frac{\sum_{k \in E} \frac{\gamma(k)}{T(a, k; 0)} T(a, k; c)}{\sum_{k \in E} \frac{\gamma(k)}{T(a, k; 0)}} \text{ and } U^\gamma(c; b) = \frac{\frac{\gamma(b)}{T(0, b; 0)} T(0, b; c)}{\sum_{k \in E} \frac{\gamma(k)}{T(0, k; 0)} T(0, k; c)} \text{ for any } a, b, c \in$$

E where γ is an eigenvector of an explicit matrix that depends only of T . This theorem is an extension of Theorem 3 of [2] valid only for 2 letters alphabet.

Inspired by this recent work, we investigate in this paper the case where the alphabet E is general (finite or infinite, discrete or not). As we have to define probability distributions on E , as usual in probability theory, we will assume that E is a Polish space (a separable complete metrizable space) equipped with its Borel set $\mathcal{B}(E)$. It could be finite or infinite and discrete or not. In the following, when we write “general alphabet”, we are thinking about a Polish space alphabet.

Let us, first, define formally a PCA with a general alphabet.

PCA with general alphabet

For PCA with general alphabets, transition matrices are replaced by transition kernels: let F and G be two Polish spaces, $K = (K(x; Y) : x \in F, Y \in \mathcal{B}(G))$ is a *transition kernel* (t.k.) from F to G if

1. $x \mapsto K(x; Y)$ is $\mathcal{B}(F)$ -measurable for all $Y \in \mathcal{B}(G)$,
2. $Y \mapsto K(x; Y)$ is a probability measure on $(G, \mathcal{B}(G))$ for all $x \in F$.

Definition 1.1 (Probabilistic cellular automata with a general alphabet). Let E be a Polish space, \mathbb{L} a lattice, N a neighborhood function and T a t.k. from $E^{|\mathbb{L}|}$ to E . The PCA $\mathbf{A} = (E, \mathbb{L}, N, T)$ is the dynamical system on $E^{\mathbb{L}}$ such that, for all $k \in \mathbb{N}$, for all $j_1, \dots, j_k \in \mathbb{L}$, $C_1, \dots, C_k \in \mathcal{B}(E)$, $t \in \mathbb{N}$,

$$P(S(j_1, t+1) \in C_1, \dots, S(j_k, t+1) \in C_k \mid (S(i, t) : i \in \mathbb{L})) = \prod_{l=1}^k T((S(i, t) : i \in N(j_l)); C_l).$$

If E is finite, this definition is similar to the classical definition of PCA. But, now, the alphabet E can be non-discrete and the t.k. can contain a non-atomic part.

Example 1.2 (Gaussian PCA). We define a family of PCA $(\mathbf{G}_{m,\sigma})$ on \mathbb{N} with alphabet \mathbb{R} and neighborhood $N(i) = (i, i+1)$ depending on two positive parameters m and σ . The t.k. of $\mathbf{G}_{m,\sigma}$ is the following: for all $a, b \in \mathbb{R}$ and Borel set $C \in \mathcal{B}(\mathbb{R})$,

$$T(a, b; C) = P\left(\mathcal{N}\left(\frac{a+b}{m}, \sigma^2\right) \in C\right)$$

where $\mathcal{N}\left(\frac{a+b}{m}, \sigma^2\right)$ is the Gaussian random variable with mean $\frac{a+b}{m}$ and variance σ^2 . In Section 3.2.1, we prove that an invariant measure of this PCA is related to autoregressive processes of order 1 (AR(1) processes).

PCA with infinite and non-discrete alphabets exist in the literature, even if they are not studied as such to the best knowledge of the author. For example, in Section 3.3, we will see that the synchronous TASEP on \mathbb{R} defined by Blank [3] (it is a discrete time, synchronous, space continuous version of the TASEP studied by Derrida & al. [6]) could be modeled by a PCA on \mathbb{Z} with alphabet $E = \mathbb{R}$ and neighborhood $N(i) = (i, i+1)$.

The *aim of the paper* is to shed some light on the structure of the set of PCA with a general alphabet (finite or infinite, discrete or not) having a Markovian invariant distribution on lattices \mathbb{N} , \mathbb{Z} or $\mathbb{Z}/n\mathbb{Z}$. In this case, some important complications arise due to measurability issues (compared with the finite case).

In continuous probability, it is classical that two distributions having a density are equal, if these densities are equal almost everywhere for the Lebesgue measure. This fact holds in a more general context: if μ is a σ -finite measure and ν_1 and ν_2 two measures absolutely

continuous with respect to μ , they are equal if their Radon-Nikodym-derivatives with respect to μ are equal μ -almost everywhere. But, for Markov chains with general alphabets, the equality μ -almost everywhere for transition kernel (i.e. for μ -almost x , $M(x; \cdot) = M'(x; \cdot)$) is not a sufficient condition to have equality in distribution for the Markov chains. Indeed, it also depends of the initial probability distribution ρ of these Markov chains. If ρ charges a μ -negligible set on which $M(x; \cdot) \neq M'(x; \cdot)$, then the two Markov chains (ρ, M) and (ρ, M') can be not equal in distribution, and even, live on two different sets.

For PCA with any general alphabet, the same complications arise: a unique PCA can have some “plural behaviors”. Hence, in this paper, each time a PCA \mathbf{A} is studied, a σ -finite measure μ is specified and, formally, it is on the pair (\mathbf{A}, μ) that the conditions and/or results hold.

Example 1.3 (Gaussian PCA except on the diagonal). We define a family of PCA $(\tilde{\mathbf{G}}_{m,\sigma})$ on \mathbb{N} with alphabet \mathbb{R} depending on two positives parameters m and σ . The t.k. \tilde{T} of $\tilde{\mathbf{G}}_{m,\sigma}$ is the same as for $\mathbf{G}_{m,\sigma}$ (defined in Example 1.2) except when $a = b$, in this case, for any $C \in \mathcal{B}(\mathbb{R})$, $\tilde{T}(a, a; C) = \delta_a(C)$ where δ_a is the Dirac measure in a .

The PCA $\tilde{\mathbf{G}}_{m,\sigma}$ will have the same behavior as the Gaussian PCA $\mathbf{G}_{m,\sigma}$ if the initial state S_{t_0} does not contain two consecutive cells in the same state, i.e. for any i , $S(i, t_0) \neq S(i+1, t_0)$. But, if its initial state is $0^{\mathbb{N}}$, then it will stay in this configuration until the end.

Before introducing the set of studied PCA in this article, let define some crucial notion used all along the paper: μ -supported and μ -positive transition kernels.

Notation. If μ is a measure on E and $d \in \mathbb{N}$, then μ^d denotes the product measure of d copies of the measures μ on E^d .

Definition 1.4 (μ -supported and μ -positive transition kernels). Let E be a Polish space, μ a σ -finite measure on E and $d \in \mathbb{N}$. Let K be a transition kernel from E^d to E , K is said to be μ -supported if for μ^d -almost (x_1, \dots, x_d) , $K(x_1, \dots, x_d; \cdot) \ll \mu$; if, moreover, for μ^d -almost (x_1, \dots, x_d) , $\mu \ll K(x_1, \dots, x_d; \cdot)$, then K is said to be μ -positive.

For K a μ -supported transition kernel from E^d to E , the μ -density of K is the μ^{d+1} -measurable function k such that

$$\begin{aligned} k : E^{d+1} &\longrightarrow \mathbb{R} \\ k(x_1, \dots, x_d; y) &\longmapsto \frac{dK(x_1, \dots, x_d; \cdot)}{d\mu}(y). \end{aligned}$$

If, moreover, K is μ -positive, then, for μ^{d+1} -almost (x_1, \dots, x_d, y) , $k(x_1, \dots, x_d; y) > 0$.

In the following, we will work with μ -supported or μ -positive kernels for $d = 1$ (transition kernels of Markov chain) or $d = |N| = 2$ (transition kernels of PCA).

We will see that such transition kernels permit to work with densities instead of measures. In the following, the Radon-Nikodym-derivative of any measure with respect to μ will be also shorten in μ -density.

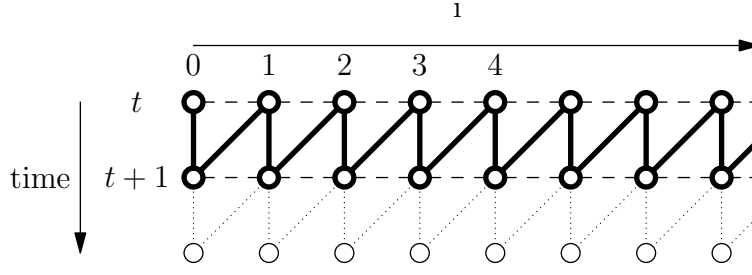


Figure 1: In bold, $\text{HZ}_{\mathbb{N}}(t)$, the t th horizontal zigzag on \mathbb{N} on a space time diagram.

An example of a Lebesgue-supported t.k. is the t.k. T of Gaussian PCA (defined in Example 1.2). This t.k. is even Lebesgue-positive. In the following, we call a μ -supported (resp. μ -positive) PCA a PCA whose t.k. is μ -supported (resp. μ -positive).

We will make apparent below (in particular in Section 3.1 and 3.2.1) that to describe the invariant distribution of a PCA, at least in the case where it admits a Markov chain as invariant distribution, that we have to work under a reference measure μ , which depending on the case can be the Lebesgue measure, a discrete measure, or any σ -finite measure. An example of that is the PCA $\tilde{\mathbf{G}}_{m,\sigma}$ of Example 1.3 for which we will find different invariant distributions according to whether the reference measure is the Lebesgue-measure or δ_a .

Remark 1.5. *There exists some transition kernels that are not μ -supported by any σ -finite measure μ . For example, the t.k. T from \mathbb{R}^2 to \mathbb{R} defined by, for any $a, b \in \mathbb{R}$, $C \in \mathcal{B}(\mathbb{R})$,*

$$T(a, b; C) = \begin{cases} \delta_a(C) & \text{if } a \neq b \\ \int_C \frac{1}{\sqrt{2\pi}} e^{-\frac{(c-a)^2}{2}} dc & \text{if } a = b \end{cases} \text{ is not } \mu\text{-supported.}$$

Indeed, any measure μ that could support this PCA has necessarily an atom at each x in \mathbb{R} . Then, μ is not a σ -finite measure.

Studied PCA in this work are the set of μ -supported PCA and its subset of μ -positive PCA. For both sets, we characterize PCA that have an invariant horizontal zigzag Markov chain, as defined now.

Let define the horizontal zigzag Markov chains (HZMC) on \mathbb{N} . First, the geometrical structure of *horizontal zigzag* is: the t th horizontal zigzag on a space-time diagram is

$$\text{HZ}_{\mathbb{N}}(t) = \left\{ \left(\left\lfloor \frac{i}{2} \right\rfloor, t + \frac{1 + (-1)^{i+1}}{2} \right), i \in \mathbb{N} \right\}$$

as illustrated in Figure 1.

Since $\text{HZ}_{\mathbb{N}}(t)$ is made by two lines corresponding to two successive times, a PCA \mathbf{A} on \mathbb{N} can be seen as acting on the configurations of $\text{HZ}_{\mathbb{N}}$. The image of a configuration $(S(i, t), S(i, t+1) : i \in \mathbb{N})$ on $\text{HZ}_{\mathbb{N}}(t)$ by the PCA \mathbf{A} is $(S(i, t+1), S(i, t+2) : i \in \mathbb{N})$ on $\text{HZ}_{\mathbb{N}}(t+1)$. Where the configuration of the second line of $\text{HZ}_{\mathbb{N}}(t)$ becomes the configuration

of the first line of $\text{HZ}_{\mathbb{N}}(t+1)$ and the configuration of the second line of $\text{HZ}_{\mathbb{N}}(t+1)$ is the image by \mathbf{A} of the second line of $\text{HZ}_{\mathbb{N}}(t)$.

Definition 1.6. An *horizontal zigzag Markov chain* (HZMC) on $\text{HZ}_{\mathbb{N}}(t)$ with general alphabet E is a Markov chain with two t.k. D (for down) and U (for up) from E to E and an initial probability distribution ρ_0 on E such that

1. the distribution of state $S(0, t)$ is ρ_0 ,
2. the distribution of state $S(i, t+1)$ knowing $S(i, t) = x_i$ is $D(x_i; \cdot)$ and
3. the distribution of state $S(i+1, t)$ knowing $S(i, t+1) = y_i$ is $U(y_i; \cdot)$.

In the following, we study under which conditions a PCA admits a HZMC as invariant distribution. For μ -supported PCA, the HZMC itself will be μ -supported: a (ρ_0, D, U) -HZMC is μ -supported if, $\rho_0 \ll \mu$ and D and U are μ -supported. In that case, we denote r_0 , d and u their respective μ -densities. Hence, a μ -supported (ρ_0, D, U) -HZMC is invariant by a μ -supported PCA with t.k. T , if, for any $k \geq 0$, for μ -almost $b_0, b_1, \dots, b_{k+1}, c_0, \dots, c_k \in E$,

$$\begin{aligned} r_0(b_0) \left(\prod_{i=0}^k d(b_i; c_i) u(c_i; b_{i+1}) \right) \\ = \int_{E^{k+3}} r_0(a_0) \left(\prod_{i=0}^{k+1} d(a_i; b_i) u(b_i; a_{i+1}) \right) \left(\prod_{i=0}^k t(b_i, b_{i+1}; c_i) \right) d\mu^{k+3}(a_0, \dots, a_{k+2}) \quad (2) \end{aligned}$$

The *support* $\tilde{E}_{(\rho_0, D, U)}$ of a (ρ_0, D, U) -HZMC on $\text{HZ}_{\mathbb{N}}(t)$ is the union of the support of the marginals of the first line of the HZMC, i.e. $\tilde{E}_{(\rho_0, D, U)} = \bigcup_{i \in \mathbb{N}} \text{supp}(\rho_i)$ where ρ_i is the distribution of $S(i, t)$. When the (ρ_0, D, U) -HZMC is μ -supported, then, for μ -almost $x \in \tilde{E}_{(\rho_0, D, U)}$, there exists $i \in \mathbb{N}$ such that $r_i(x) > 0$ (that holds because E is a Polish space). In the case of a μ -positive (ρ_0, D, U) -HZMC, $\tilde{E}_{(\rho_0, D, U)} = \text{supp}(\mu)$. When the context is clear, $\tilde{E}_{(\rho_0, D, U)}$ will be denoted \tilde{E} .

Remark 1.7. Take a μ -supported PCA \mathbf{A} with t.k. T and a μ -supported (ρ_0, D, U) -HZMC with support \tilde{E} . Suppose that the (ρ_0, D, U) -HZMC is invariant by \mathbf{A} . Now take a μ -supported PCA \mathbf{A}' with t.k. T' such that, for any $a, b \in \tilde{E}$, $T'(a, b; \cdot) = T(a, b; \cdot)$. Then, the (ρ_0, D, U) -HZMC is also invariant by \mathbf{A}' . Hence, to characterize if a HZMC with support \tilde{E} is invariant by a PCA with t.k. T , the value of $T(a, b; \cdot)$ for a or b not in \tilde{E} are not necessary.

Let μ be a measure on E and $d : (a, c) \mapsto d(a; c)$ and $u : (c, b) \mapsto u(c; b)$ be two μ^2 -measurable functions from E^2 to \mathbb{R} , then the μ^2 -measurable function \overline{du} from E^2 to \mathbb{R} is defined by $\overline{du}(a; b) = \int_E d(a; c) u(c; b) d\mu(c)$. For a μ -supported HZMC, $\overline{du}(a; b)$ is the μ -density of the t.k. (DU) of the Markov chain (induced by the HZMC) on the first line $S_t = (S(i, t) : i \in \mathbb{N})$ of $\text{HZ}_{\mathbb{N}}(t)$.

Main results

We start with a generalization to Polish space alphabets of Lemma 2.3 in [5].

Theorem 1.8. *Let μ be a σ -finite measure on a general alphabet E . Let $\mathbf{A} := (\mathbb{N}, E, N, T)$ be a μ -supported PCA and (ρ_0, D, U) a μ -supported HZMC with support \tilde{E} . The (ρ_0, D, U) -HZMC is invariant by \mathbf{A} if and only if the three following conditions are satisfied:*

Cond 1: *for μ^3 -almost $(a, b, c) \in \tilde{E}^3$, $t(a, b; c)\overline{d}u(a; b) = d(a; c)u(c; b)$,*

Cond 2: *for μ^2 -almost $(a, b) \in \tilde{E}^2$, $\overline{d}u(a; b) = \overline{u}d(a; b)$,*

Cond 3: *the Markov chain with t.k. D possesses ρ_0 as invariant distribution, i.e. for μ -almost c , $r_0(c) = \int_E r_0(a)d(a; c)d\mu(a)$.*

We arrive to our main Theorem 1.9. When a PCA with t.k. T is μ -positive, we can go further and reduce the existence of an invariant HZMC for the PCA to the existence of a function η solution to a cubic integral equation on T . In case of existence, we can express the kernels of the invariant HZMC using η and T . Let us first introduce some material.

Let \mathbf{A} be a PCA with t.k. T whose μ -density is t . Define, for any positive measurable function $\phi \in L^1(\mu)$ (i.e. for μ -almost $x \in E$, $\phi(x) > 0$ and $\int_E \phi(x)d\mu(x) < \infty$), the two μ^2 -measurable functions $d^\phi : E^2 \mapsto \mathbb{R}$ and $u^\phi : E^2 \mapsto \mathbb{R}$ by

$$d^\phi(a; c) = \frac{\int_E \frac{\phi(x)}{t(a, x; c_0)} t(a, x; c) d\mu(x)}{\int_E \frac{\phi(x)}{t(a, x; c_0)} d\mu(x)} \text{ and } u^\phi(c; b) = \frac{\frac{\phi(b)}{t(a_0, b; c_0)} t(a_0, b; c)}{\int_E \frac{\phi(x)}{t(a_0, x; c_0)} t(a_0, x; c) d\mu(x)}. \quad (3)$$

Theorem 1.9. *Let μ be a σ -finite measure on a general alphabet E . Let $\mathbf{A} := (\mathbb{N}, E, N, T)$ be a μ -positive PCA. \mathbf{A} admits a μ -positive invariant HZMC if and only if the three following conditions are satisfied:*

Cond 4: *there exists a triplet $(a_0, b_0, c_0) \in E^3$ such that $T(a_0, b_0; \cdot)$ and μ are positive equivalent and, for μ^3 -almost (a, b, c) ,*

$$t(a, b; c)t(a_0, b_0; c)t(a_0, b; c_0)t(a, b_0; c_0) = t(a_0, b_0; c_0)t(a, b; c_0)t(a, b_0; c)t(a_0, b; c),$$

Cond 5: *there exists a positive function $\eta \in L^1(\mu)$ solution to: for μ^2 -almost (a, b) and for the (a_0, c_0) of **Cond 4**,*

$$\frac{\frac{\eta(b)}{t(a, b; c_0)}}{\int_E \frac{\eta(x)}{t(a, x; c_0)} d\mu(x)} = \int_E \frac{\frac{\eta(c)}{t(a_0, c; c_0)} t(a_0, c; a)}{\int_E \frac{\eta(x)}{t(c, x; c_0)} d\mu(x)} \frac{\int_E \frac{\eta(x)}{t(c, x; c_0)} t(c, x; b) d\mu(x)}{\int_E \frac{\eta(x)}{t(a_0, x; c_0)} t(a_0, x; a) d\mu(x)} d\mu(c), \quad (4)$$

Cond 6: *the Markov chain with t.k. D^η , whose μ -density is d^η given by Eq (3), possesses as invariant distribution a probability distribution ρ_0 .*

In this case, the (ρ_0, D^η, U^η) -HZMC where D^η and U^η are t.k. of μ -densities given by Eq (3) is invariant by \mathbf{A} .

Remark 1.10. If **Cond 4** and **Cond 5** hold and if E is finite, the Markov chain with t.k. D^n is irreducible and aperiodic (because, for any $a, c \in E$, $D^n(a, c) > 0$) and, so, it possesses a unique invariant distribution, i.e. **Cond 6** always holds. If E is not finite, we refer the reader to the book of Meyn and Tweedie [11] to get some conditions on D^n for which the Markov chain with t.k. D^n possesses an invariant distribution.

When the alphabet is finite, we can go further and show that if η satisfies Eq (4) then η is the eigenvector of a computable matrix, obtaining such a way a simple and strong condition for the existence of such η (this is done in [5]). For PCA with a general alphabet, this can not be done due to measurability issues that, roughly, do not allow us to take $a = b$ in Eq (4). Nevertheless, under stronger conditions on t , we can characterize a set of functions that contains the set of functions η solution to Eq (4).

Proposition 1.11. Let μ be a σ -finite measure on a general alphabet E . Let $\mathbf{A} := (\mathbb{Z}, E, N, T)$ be a μ -positive PCA. Suppose that **Cond 4** and the two following conditions are satisfied:

Cond 7: for the same triplet (a_0, b_0, c_0) of **Cond 4**, for μ^2 -almost (a, c) ,

$$t(a, a; c)t(a_0, b_0; c)t(a_0, a; c_0)t(a, b_0; c_0) = t(a_0, b_0; c_0)t(a, a; c_0)t(a, b_0; c)t(a_0, a; c), \quad (5)$$

Cond 8: there exists a positive function $\eta \in L^1(\mu)$ solution to: for μ -almost a and for the (a_0, c_0) of **Cond 4**,

$$\frac{\frac{\eta(a)}{t(a, a; c_0)}}{\int_E \frac{\eta(k)}{t(a, x; c_0)} d\mu(x)} = \int_E \frac{\frac{\eta(c)}{t(a_0, c; c_0)} t(a_0, c; a)}{\int_E \frac{\eta(x)}{t(c, x; c_0)} d\mu(x)} \frac{\int_E \frac{\eta(x)}{t(c, x; c_0)} t(c, x; a) d\mu(x)}{\int_E \frac{\eta(x)}{t(a_0, x; c_0)} t(a_0, x; a) d\mu(x)} d\mu(c). \quad (6)$$

Then, η is a positive eigenfunction of

$$\mathcal{A}_2 : f \mapsto \left(\mathcal{A}_2(f) : a \mapsto \int_E f(k) \frac{t(a, a; c_0)}{t(a, x; c_0)} \nu(a) d\mu(x) \right)$$

where ν is a positive eigenfunction (unique up to a multiplicative constant) in $L^1(\mu)$ of

$$\mathcal{A}_1 : f \mapsto \left(\mathcal{A}_1(f) : a \mapsto \int f(c) t(c, c; a) d\mu(c) \right).$$

Remark 1.12. • Any positive PCA with finite alphabet E (i.e. for all a, b, c , $T(a, b; c) > 0$) is a μ_E -positive PCA where μ_E is the counting measure on E . Hence, **Cond 7** and **Cond 8** are necessary implied by **Cond 4** and **Cond 5** in the case of finite alphabets. Moreover, in that case, \mathcal{A}_1 and \mathcal{A}_2 have their own unique eigenfunction (due to Perron-Frobenius theorem) and **Cond 6** holds necessarily. So, applying Theorem 1.9 and Prop 1.11 to positive PCA give Theorem 2.6 in [5].

• Let $E = \mathbb{R}$ and μ be the Lebesgue measure. In the case where t is continuous at any point of E^3 , then **Cond 4** and **Cond 5** imply **Cond 7** and **Cond 8** by continuity. And so a solution η to Eq (4) is a function η given by Prop 1.11.

• If for a PCA \mathbf{A} the conditions of Prop 1.11 do not hold, it is in general complex to find a function η solution to Eq (4). But, it may happen that a μ -equivalent PCA \mathbf{A}' to \mathbf{A} (see Definition 2.3 in Section 2) satisfies the conditions of Prop 1.11. Hence, in the best-case scenario, we can characterize, thanks to Prop 1.11, a (ρ_0, D^n, U^n) -HZMC invariant by \mathbf{A}' . And, by Prop 2.4 (in Section 2), this HZMC is also invariant by \mathbf{A} . An application of this method is shown in Section 3.2.1 where it is proved that AR(1) process is an invariant distribution of $\tilde{\mathbf{G}}_{m,\sigma}$ (defined in Example 1.3).

The uniqueness (up to a multiplicative constant) of the eigenfunction ν (in Prop 1.11) is a consequence of the following lemma.

Lemma 1.13 (Theorem 6.8.7 of Durrett [7]). *Let $\mathcal{A} : f \mapsto \left(\mathcal{A}(f) : y \mapsto \int_E f(x)m(x;y)\mu(dx) \right)$ be an integral operator of kernel m . If m is the μ -density of a μ -positive t.k. M from E to E , then \mathcal{A} possesses at most one positive eigenfunction in $L^1(\mu)$ (up to a multiplicative constant).*

Content

In Section 2, we recall some facts about Radon-Nikodym theorem and, then, state some properties of μ -supported and μ -positive PCA.

Section 3 is dedicated to some examples of PCA. In Section 3.1, we show applications of Theorems 1.8 and 1.9 and Prop 1.11 to PCA with finite alphabets. In Section 3.2.1, we use Theorem 1.9 and Prop 1.11 to show that the law of an autoregressive process of order 1 (AR(1) process) is invariant by both Gaussian PCA $\mathbf{G}_{m,\sigma}$ and $\tilde{\mathbf{G}}_{m,\sigma}$ (defined in Example 1.2 and 1.3). In Section 3.2.2, we present a Lebesgue-supported PCA called Beta PCA. In Section 3.3, we present first a PCA with alphabet \mathbb{R} that simulates a synchronous TASEP on \mathbb{R} as defined by Blank [3] and, then, a PCA with alphabet \mathbb{R} that simulates the first-passage percolation as presented by Kesten [9] on a particular graph \mathcal{G} . Unfortunately, Theorem 1.8 and 1.9 do not apply to these two PCA.

In Section 4, Theorems 1.8 and 1.9 and Prop 1.11, the main contributions of the paper, are proved.

Section 5 is devoted to extensions of Theorems 1.8 and 1.9 for PCA on \mathbb{Z} and $\mathbb{Z}/n\mathbb{Z}$. First, we extend in both cases the notion of HZMC: HZMC $_{\mathbb{Z}}$ on \mathbb{Z} and cyclic-HZMC (CHZMC) on $\mathbb{Z}/n\mathbb{Z}$ (if E is finite, a CHZMC is an HZMC conditioned to be periodic and, in the general case, it is a Gibbs measure). Then, we characterize PCA letting HZMC $_{\mathbb{Z}}$ invariant, and also PCA letting CHZMC invariant.

2 Preliminaries

We recall here some facts around Radon-Nikodym theorem.

Let μ and ν be two measures on E . μ is *equal* to ν ($\mu = \nu$) if for all $A \in \mathcal{B}(E)$, $\mu(A) = \nu(A)$. μ is *absolutely continuous* with respect to ν ($\mu \ll \nu$) if, for all $A \in \mathcal{B}(E)$, $\nu(A) = 0 \Rightarrow \mu(A) = 0$. And μ and ν are *singular* ($\mu \perp \nu$) if there exists $N \in \mathcal{B}(E)$ such that $\mu(N) = 0$ and $\nu(N^c) = 0$. The *Radon-Nikodym theorem* allows one to decompose a σ -finite measure with respect to another. Let ν, μ be two positive σ -finite measures on E , then there exists a unique pair of positive σ -finite measures (μ_1, μ_2) such that $\mu = \mu_1 + \mu_2$ with $\mu_1 \ll \nu$ and $\mu_2 \perp \nu$. Moreover, there exists a unique (up to a ν -null set) ν -measurable function $f : E \rightarrow \mathbb{R}^+$ such that, for all $A \in \mathcal{B}(E)$, $\mu_1(A) = \int_A f d\nu$. The function f is denoted $\frac{d\mu}{d\nu}$ and called Radon-Nikodym-derivative of μ with respect to ν (or ν -density).

Definition 2.1 (Positive equivalence). Let ν, μ be two measures on E . ν and μ are *positive equivalent* if $\nu \ll \mu$ and $\mu \ll \nu$. In that case, $\frac{d\mu}{d\nu} > 0$ and $\frac{d\nu}{d\mu} > 0$, μ -almost everywhere.

Now, we give some properties of μ -positive PCA and define the μ -equivalence of PCA.

Proposition 2.2. *Let \mathbf{A} be a PCA. If \mathbf{A} is μ and ν -positive, then μ and ν are positive equivalent or singular.*

Proof. Let \mathbf{A} be a PCA that is both μ and ν -positive. If there exists $(a, b) \in E^2$ such that the measure $T(a, b; \cdot)$ is both μ and ν -positive then ν is μ -positive by transitivity. Else, $P_\mu = \{(a, b) : T(a, b; \cdot) \text{ is } \mu\text{-positive}\}$ and $P_\nu = \{(a, b) : T(a, b; \cdot) \text{ is } \nu\text{-positive}\}$ are measurable and disjoint, and so taking $N = P_\nu \subset P_\mu^c$, $\mu(N) = 0$ and $\nu(N^c) = 0$. \square

The PCA $(\tilde{\mathbf{G}}_{m,\sigma})$ (defined in Example 1.3) are Lebesgue-positive; indeed, $\{(a, b) : T(a, b; \cdot) \ll \text{Lebesgue-measure}\} = \{(a, a) : a \in \mathbb{R}\}$ is Lebesgue-negligible in \mathbb{R}^2 . Moreover, for any $a \in \mathbb{R}$, they are δ_a -measurable because $T(a, a; \cdot) = \delta_a$. One can verify that Prop 2.2 holds for these PCA because $\{\delta_a : a \in \mathbb{R}\}$ and the Lebesgue-measure are pairwise singular.

Definition 2.3 (μ -equivalent PCA). Let \mathbf{A} and \mathbf{A}' be two μ -supported PCA with respective t.k. T and T' . \mathbf{A} and \mathbf{A}' are said to be μ -*equivalent* if the set where T and T' are not equal is a μ^2 -negligible set, i.e. $\mu^2(\{(a, b) : T(a, b; \cdot) \neq T'(a, b; \cdot)\}) = 0$.

For any (m, σ) , the Gaussian PCA $\mathbf{G}_{m,\sigma}$ (defined in Example 1.2) and the PCA $\tilde{\mathbf{G}}_{m,\sigma}$ are Lebesgue-equivalent (their t.k. differs on $\{(a, a) : a \in \mathbb{R}\}$, a Lebesgue-negligible set).

Proposition 2.4. *Let \mathbf{A} and \mathbf{A}' be two μ -equivalent PCA and (ρ_0, D, U) a μ -supported HZMC. If (ρ_0, D, U) is an invariant measure for \mathbf{A} , then (ρ_0, D, U) is also an invariant measure for \mathbf{A}' .*

Proof. By property of μ -equivalent PCA, we can change t by t' in Eq (2). \square

Hence, sometimes, to find an invariant HZMC of a μ -supported PCA \mathbf{A} , the easiest way is to find a μ -equivalent PCA \mathbf{A}' for which we already know a μ -positive invariant HZMC.

In particular, for a μ -positive PCA \mathbf{A} for which Prop 1.11 does not apply, it could exist a μ -equivalent PCA \mathbf{A}' for which this Proposition applies and gives a solution η to Eq (4). This Proposition gives some “degrees of freedom” on the “rigid” integral cubic equation Eq (4). In Section 3.2.1, this Proposition will be used to prove that an invariant measure to $\tilde{\mathbf{G}}_{m,\sigma}$ is an AR(1) process.

3 Examples

Notation. In this section, if E is a finite set, then $\mu_E = \sum_{x \in E} \delta_x$ is the counting measure on E .

Our first examples are PCA with finite alphabets. Then, we introduce two new models: Gaussian PCA and Beta PCA to illustrate our theorems. Finally, we present PCA with infinite alphabets that model existing problems in literature: one PCA models a synchronous TASEP on \mathbb{R} as defined by Blank [3] and an other one a variant of directed first-passage percolation.

All PCA presented in this section are PCA on \mathbb{N} (except the PCA modeling TASEP that is on \mathbb{Z}) and neighborhood $N(i) = (i, i + 1)$.

3.1 PCA with finite alphabet

For positive PCA, see the first point of Remark 1.12.

In the following example, we focus on PCA that are not positive and take a PCA not μ_E -positive, but μ_F -positive for some F subsets of E .

Let \mathbf{A} be the PCA with alphabet $E = \{0, 1, 2\}$ and t.k.:

- $T(0, i; i) = T(i, 0; i) = 1$ for all $i \in \{0, 1, 2\}$,
- $T(1, 1; 1) = T(1, 1; 2) = T(2, 2; 1) = T(2, 2; 2) = 1/2$,
- $T(1, 2; 1) = T(2, 1; 2) = 4/5$,
- $T(1, 2; 2) = T(2, 1; 1) = 1/5$.

This PCA is not positive ($T(0, 1; 0) = 0$), nevertheless it is $\mu_{\{0\}}$ -positive ($T(0, 0; \cdot) = \mu_{\{0\}}(\cdot)$) and, also, $\mu_{\{1,2\}}$ -positive. These two measures are singular as “predicted” by Prop 2.2.

Considered as a $\mu_{\{0\}}$ -positive PCA, Theorem 5.2 and Prop 1.11 to \mathbf{A} imply that the constant (equals to 0) HZMC is invariant by \mathbf{A} .

Application of the same Theorem and same Lemma when \mathbf{A} is considered as a $\mu_{\{1,2\}}$ -positive PCA gives: that **Cond** 4 holds because

$$T(1, 1; 1)T(2, 2; 1)T(1, 2; 2)T(2, 1; 2) = T(2, 2; 2)T(1, 1; 2)T(2, 1; 1)T(1, 2; 1) = 1/25,$$

for (a_0, b_0, c_0) we take $(1, 1, 1)$; then we obtain for ν and η , as defined in Prop 1.11, $\nu(1) = \nu(2) = 1/2$, $\eta(1) = 1/3$ and $\eta(2) = 2/3$ and, so, taking D^η and U^η as defined in Eq (3): $D^\eta(1; 1) = D^\eta(2; 2) = 2/3$, $D^\eta(1; 2) = D^\eta(2; 1) = 1/3$, $U^\eta(1; 1) = U^\eta(2; 2) = 1/3$ and

$U^n(1;2) = U^n(2;1) = 2/3$. Then, $\rho_0(1) = \rho_0(2) = 1/2$ is an invariant measure for the Markov chain of kernel D^n . Hence, there are two HZMC, supported by two singular measures $(\mu_{\{0\}}$ and $\mu_{\{1,2\}}$), invariant by \mathbf{A} .

A μ -supported PCA

Let \mathbf{A} be the PCA with alphabet $E = \mathbb{Z}/\kappa\mathbb{Z}$ with t.k. T such that $T(a, b; \cdot)$ is the uniform distribution on the circular interval set $\{a+1, \dots, b-1\}$ if $|a-b| > 1$ and if $|a-b| \leq 1$, it is the uniform distribution on E .

This PCA is a μ_E -supported PCA, but not μ -positive for any measure μ on E . This PCA \mathbf{A} has an invariant (ρ_0, D, U) HZMC with $D(a; a+1 \bmod \kappa) = U(a; a+1 \bmod \kappa) = 1$ for all $a \in \mathbb{Z}/\kappa\mathbb{Z}$ and for any $a \in \mathbb{Z}/\kappa\mathbb{Z}$, $\rho_0(\kappa) = \frac{1}{\kappa}$.

3.2 Two new models of PCA with infinite alphabet

3.2.1 Gaussian PCA

Notation. In the following, for any two positive parameters m and σ , the Lebesgue-density of the Gaussian distribution of mean m and variance σ^2 will be denoted

$$g[m, \sigma](x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right).$$

In this section, we apply Theorem 1.9 and Prop 1.11 to prove that an AR(1) process is an invariant distribution for Gaussian PCA $\mathbf{G}_{m,\sigma}$ (defined in Example 1.2). Then, we prove the same property for PCA $\tilde{\mathbf{G}}_{m,\sigma}$ (defined in Example 1.3) by an application of Prop 2.4.

Gaussian PCA $\mathbf{G}_{m,\sigma}$. For $\mathbf{G}_{m,\sigma}$, it can be checked that **Cond** 4 holds for any triplet (a_0, b_0, c_0) in \mathbb{R}^3 , so let us choose $(a_0, b_0, c_0) = (0, 0, 0)$. We use Prop 1.11 to obtain a function η . The first step consists in studying the eigenfunctions of

$$\begin{aligned} \mathcal{A}_1 : L^1 &\longrightarrow L^1 \\ f &\longmapsto \mathcal{A}_1(f) : c \mapsto \int_{\mathbb{R}} f(a) g\left[\frac{2}{m}a, \sigma\right](c) da. \end{aligned}$$

The function $\nu(x) = \exp\left(-\frac{1}{2\sigma^2}\left(1 - \frac{4}{m^2}\right)x^2\right)$ is a positive eigenfunction of \mathcal{A}_1 . Moreover, we need ν to be in L^1 , hence $1 - \frac{4}{m^2}$ must be positive and, so, we need $|m| > 2$. Without this condition, for any i , the function $t \rightarrow \text{Var}(S(i, t))$ increases and goes to infinity with t . When $|m| > 2$, we can go further with Prop 1.11 and study the eigenfunctions of

$$\begin{aligned} \mathcal{A}_2 : L^1 &\longrightarrow L^1 \\ f &\longmapsto \mathcal{A}_2(f) : a \mapsto \int_{\mathbb{R}} f(b) \frac{t(a, a; 0)\nu(a)}{t(a, b; 0)} db \end{aligned}$$

with $\frac{t(a, a; 0)\nu(a)}{t(a, b; 0)} = \exp\left(-\frac{b^2}{2\sigma^2}\right) \exp\left(\frac{\left(\frac{a+b}{m}\right)^2}{2\sigma^2}\right)$. One can check that the function

$$\eta(x) = \exp\left(-\frac{1}{4\sigma^2} \left(1 + \sqrt{1 - \frac{4}{m^2}}\right) x^2\right)$$

is a positive eigenfunction of \mathcal{A}_2 associated to the eigenvalue $\frac{\sqrt{\pi\sigma^2}}{\left(1 + \sqrt{1 - \frac{4}{m^2}}\right)^2}$. Moreover η satisfies Eq (4) (this is an example where Prop 1.11 permits to compute a solution η to Eq (4)). We get

$$d^\eta(a; c) = g\left[\frac{2}{ml} a, \sqrt{\frac{2}{l}} \sigma\right](c) \quad (7)$$

and

$$u^\eta(c; b) = g\left[\frac{2}{ml} c, \sqrt{\frac{2}{l}} \sigma\right](b) \quad (8)$$

for $l = 1 + \sqrt{1 - \frac{4}{m^2}}$. To end, we need to find an invariant probability distribution ρ_0 for the Markov chain of t.k. D^η (of Lebesgue-density d^η). The measure ρ_0 with the following Lebesgue-density r_0 is fine:

$$r_0(x) = g\left[0, \left(1 - \frac{4}{m^2}\right)^{-1/4} \sigma\right](x) \quad (9)$$

This permits to conclude that the (ρ_0, D^η, U^η) -HZMC is an invariant measure for the Gaussian PCA.

In fact, this invariant HZMC is an autoregressive process of order 1 (AR(1) process, see [14]) that is a process (X_i) such that $X_i = \theta + \phi X_{i-1} + \epsilon_i$ where θ and ϕ are two real numbers and (ϵ_i) are independent and identically distributed of law the Gaussian law $\mathcal{N}(0, \sigma'^2)$. In our case, the invariant HZMC is an AR(1) process on $\text{HZ}_{\mathbb{N}}$ with $\theta = 0$, $\phi = \frac{2}{ml}$ and $\sigma'^2 = \frac{2\sigma^2}{l}$.

“Gaussian PCA except on diagonal” $\tilde{\mathbf{G}}_{m,\sigma}$. As already seen in Section 2, this PCA is Lebesgue-positive and also $\mu_{\{a\}}$ -positive for any $a \in \mathbb{R}$.

When we consider $\tilde{\mathbf{G}}_{m,\sigma}$ as a Lebesgue-positive PCA, Prop 1.11 could not be used to find a solution η to Eq (4). Hopefully, $\tilde{\mathbf{G}}_{m,\sigma}$ is Lebesgue-equivalent to $\mathbf{G}_{m,\sigma}$. Hence, by Prop 2.4, the invariant Lebesgue-positive (ρ_0, D^η, U^η) -HZMC, that corresponds to an AR(1) process, obtained for $\mathbf{G}_{m,\sigma}$ is also invariant for $\tilde{\mathbf{G}}_{m,\sigma}$.

Besides, for any $a \in \mathbb{R}$, the constant process equal to a everywhere is also an invariant measure to $\tilde{\mathbf{G}}_{m,\sigma}$.

3.2.2 Beta PCA

We define a class of PCA with alphabet \mathbb{R} depending on three positive real parameters α , β and m . The t.k. is the following: for all $a, b \in \mathbb{R}$ and $C \in \mathcal{B}(\mathbb{R})$,

$$T(a, b; C) = P((b - a)X + a - m \in C)$$

where X is a $\text{Beta}(\alpha, \beta)$ random variable, i.e. the Lebesgue-density of T is, for μ -almost a, b, c ,

$$t(a, b; c) = \frac{\left(\frac{c + m - a}{b - a}\right)^{\alpha-1} \left(\frac{b - c - m}{b - a}\right)^{\beta-1}}{B(\alpha, \beta)} \mathbf{1}_{0 \leq \frac{c+m-a}{b-a} \leq 1}$$

where B is the beta function. In words, the PCA takes a random (following a Beta law) number between the two values of its two neighbors and subtract m to it.

This PCA is Lebesgue-supported, but not Lebesgue-positive.

Now, try to search an invariant (ρ_0, D, U) -HZMC to this PCA. Let θ be a positive real number. Let $D_1(a; C) = P(X_1 + a - m \in C)$ and $U_1(c; B) = P(X_2 + c + m \in B)$ where X_1 (resp. X_2) is a $\text{Gamma}(\alpha, \theta)$ (resp. $\text{Gamma}(\beta, \theta)$) random variable. For $D = D_1$ and $U = U_1$, **Cond 1** and **Cond 2** hold; unfortunately, there does not exist a probability distribution ρ_0 that satisfies **Cond 3**. Hence, this PCA does not possess a Lebesgue-supported HZMC as invariant distribution. Nevertheless, the image of a Lebesgue-supported (ρ, D_1, U_1) -HZMC by this PCA is the $(\rho D_1, D_1, U_1)$ -HZMC (meaning that one can describe simply the distribution of the successive image of a (ρ, D_1, U_1) -HZMC by A).

3.3 PCA with infinite alphabet in the literature

PCA modeling TASEP

We model the synchronous TASEP on \mathbb{R} introduced by Blank [3] by a PCA on \mathbb{Z} with alphabet \mathbb{R} . In the following, when we say TASEP, we refer to this variant of TASEP.

TASEP models the behavior of an infinite number of particles of radius $r \geq 0$ on the real line, that move to the right direction, that do not bypass, not overlap and, at each step of time, each particle moves with probability p ($0 < p \leq 1$), independently of each others. When a particle moves, it travels a distance $v \geq 0$ to the right direction, except if it can create a collision with the next particle, in that case, it moves to the rightmost allowed position. Formally, the evolution of (x_i^t) is the following:

$$x_i^{t+1} = \begin{cases} \min(x_i^t + v, x_{i+1}^t - 2r) & \text{with probability } p, \\ x_i^t & \text{with probability } 1 - p. \end{cases}$$

We propose, here, to model this TASEP by a PCA **A** on \mathbb{Z} with alphabet \mathbb{R} . In this model, the state of a cell i at time t is the position x_i^t of the i th particle of the TASEP at time t .

Hence, the t.k. of the PCA is the following: for any $a, b \in \mathbb{R}$ such that $a + r \leq b - r$ and for any $C \in \mathcal{B}(\mathbb{R})$,

$$T(a, b; C) = \begin{cases} (1-p)\delta_a(C) + p\delta_{a+v}(C) & \text{if } a + v \leq b - 2r, \\ (1-p)\delta_a(C) + p\delta_{b-2r}(C) & \text{if } a + v > b - 2r. \end{cases}$$

The t.k. for other pairs (a, b) is not specified since they concern forbidden configurations. Hence, if we start with an admissible configuration at time 0 for the PCA (i.e. for any $i \in \mathbb{Z}$, $S(i, 0) + r \leq S(i + 1, 0) - r$), then the PCA models the TASEP.

We can remark that if, at some time t , $v = 2kr$ for some $k \in \mathbb{N}$, and, for any i , $x_i(t) \in 2r\mathbb{Z}$, then at time $t + 1$ this is also the case. In terms of PCA, this says that the PCA \mathbf{A} is μ -supported by $\mu = \sum_{i \in \mathbb{Z}} \delta_{2ri}$. For this measure, one can check that the (R, D, U) -HZMC $_{\mathbb{Z}}$ (HZMC on \mathbb{Z} are defined in Section 5.1) where $R_i = \delta_{2ri}$, $D(a; a) = 1$ and $U(a; a + 2r) = 1$ (i.e. the states of the HZMC $_{\mathbb{Z}}$ are $S(i, t) = S(i, t + 1) = 2ri$ for all $i \in \mathbb{Z}$) is an invariant HZMC $_{\mathbb{Z}}$ for the PCA. But, it is quite an uninteresting invariant measure because it corresponds to a trivial configuration where nobody can move.

PCA modeling a variant of first-passage percolation

We propose a model of a directed first-passage percolation P on a directed graph \mathcal{G} which can also be seen as a PCA with alphabet $[0, \infty)$. We use the same notation as Kesten [9] to present the classical model of first-passage percolation.

The set of nodes of \mathcal{G} is $\mathcal{N} = \{(i, j) : i, j \in \mathbb{N}\}$, the discrete quarter plan, and the set of directed edges is $\mathcal{E} = \{((i, j), (i, j + 1)) : i, j \in \mathbb{N}\} \cup \{((i + 1, j), (i, j + 1)) : i, j \in \mathbb{N}\}$. We denote L_0 the set of the nodes of the first line $L_0 = \{(i, 0) : i \in \mathbb{N}\}$. Now, assign to each edge $e \in \mathcal{E}$ a random non-negative weight $t(e)$ which could be interpreted as the time needed to pass through the edge e . We assume that $(t(e) : e \in \mathcal{E})$ are i.i.d. with common distribution F . The passage time of a directed path $r = (e_1, \dots, e_n)$ on G is $T(r) = \sum_{i=1}^n t(e_i)$. The travel time from a node u to a node v is defined as $T(u, v) = \inf\{T(r) : r \text{ is a directed path from } u \text{ to } v\}$. If there is no directed path from u to v , $T(u, v) = \infty$. We define the travel time from a set of nodes U to a node v by $T(U, v) = \inf\{T(u, v) : u \in U\}$. Finally, we define $\mathcal{V}(t) = \{v \in \mathcal{N} : T(L_0, v) \leq t\}$ the set of visited nodes at time t . The object of study in the first-passage percolation is this set $\mathcal{V}(t)$.

The first-passage percolation P on \mathcal{G} can be seen as a PCA \mathbf{A} on \mathbb{N} with alphabet $[0, \infty)$ as follows: let $S(i, j)$ represents the travel time $T(L_0, (i, j))$ from L_0 to the node (i, j) in the first-passage percolation. Hence, the t.k. of the PCA is the following: for any $a, b \in [0, \infty)$, for any $C \in \mathcal{B}([0, \infty))$,

$$T(a, b; C) = L_{a,b}(C)$$

where $L_{a,b}$ is the distribution of the random variable $X = \min\{(a + T_1), (b + T_2)\}$ where T_1 and T_2 are i.i.d. with common law F .

Unfortunately, our work does not apply to these examples.

4 Proofs of the main results

4.1 Proof of Theorem 1.8

First: let (ρ_0, D, U) be a μ -supported HZMC invariant by \mathbf{A} with t.k. T , a μ -supported PCA. For all $A, B, C \in \mathcal{B}(E)$, for all $i \in \mathbb{N}$,

$$\begin{aligned} P(S(i, t) \in A, S(i+1, t) \in B, S(i, t+1) \in C) &= \int_{A \times B \times C} r_i(a) d(a; c) u(c; b) d\mu^3(a, b, c) \\ &= \int_{A \times B \times C} r_i(a) \overline{du}(a; b) t(a, b; c) d\mu^3(a, b, c) \end{aligned}$$

where ρ_i is the law of cell i of μ -density r_i . Taking the difference, we obtain, for all $A, B, C \in \mathcal{B}(E)$,

$$\int_{A \times B \times C} \left(r_i(a) d(a; c) u(c; b) - r_i(a) \overline{du}(a; b) t(a, b; c) \right) d\mu^3(a, b, c) = 0.$$

Hence, since this holds for any Borel set $A \times B \times C$, $r_i(a) d(a; c) u(c; b) = r_i(a) \overline{du}(a; b) t(a, b; c)$ for μ^3 -almost $(a, b, c) \in E^3$. If $a \in \tilde{E}$, there exists i such that $r_i(a) > 0$ a.s. and, then, **Cond 1** holds.

We have also, for all $A, B \in \mathcal{B}(E)$, on one hand,

$$\begin{aligned} P(S(i, t+1) \in A, S(i+1, t+1) \in B) &= P(S(i, t+1) \in A, S(i+1, t+1) \in B, S(i+1, t) \in E) \\ &= \int_{A \times B} r_i(a) \overline{ud}(a; b) d\mu^2(a, b) \end{aligned}$$

because $(S(0, t), S(0, t+1), S(1, t), \dots)$ is a (ρ_0, D, U) -HZMC and, on the other hand,

$$\begin{aligned} P(S(i, t+1) \in A, S(i+1, t+1) \in B) &= P(S(i, t+1) \in A, S(i, t+1) \in B, S(i, t+2) \in E) \\ &= \int_{A \times B} r_i(a) \overline{du}(a; b) d\mu^2(a, b) \end{aligned}$$

because $(S(0, t+1), S(0, t+2), S(1, t+1), \dots)$ is also a (ρ_0, D, U) -HZMC due to its invariance by \mathbf{A} . Then, as before, $r_i(a) \overline{ud}(a; b) = r_i(a) \overline{du}(a; b)$ for μ^2 -almost $(a, b) \in E^2$ and, so, **Cond 2** holds.

Moreover, the law of $S(0, t)$ and $S(0, t+1)$ must be the same because (ρ_0, D, U) is invariant by the PCA. Hence, the law of $S(0, t+1)$ of μ -density $\int_E r_0(a) d(a; c) d\mu(a)$ must be equal to ρ_0 of μ -density $r_0(c)$, i.e. **Cond 3** holds.

Conversely, suppose that **Cond 1**, **Cond 2** and **Cond 3** are satisfied. Suppose that the horizontal zigzag $\text{HZ}_{\mathbb{N}}(t)$ is distributed as a (ρ_0, D, U) -HZMC. Now, compute the push forward

measure of this HZMC by **A**. For any $n \geq 0$, for any $F_{2n+1} = B_0 \times \cdots \times B_{n+1} \times C_0 \times \cdots \times C_n \in \mathcal{B}(E)^{2n+1}$.

$$\begin{aligned}
& P(S(0, t+1) \in B_0, S(0, t+2) \in C_0, \dots, S(n+1, t+1) \in B_{n+1}) \\
&= \int_{E^{n+2} \times F_{2n+1}} r_0(a_0) \prod_{i=0}^{n+1} d(a_i; b_i) u(b_i; a_{i+1}) t(b_i, b_{i+1}; c_i) \\
&\quad d\mu^{3n+6}(a_0, \dots, a_{n+2}, b_0, \dots, b_{n+1}, c_0, \dots, c_n) \\
&= \int_{F_{2n+1}} \left(\int_E r_0(a_0) d(a_0; b_0) d\mu(a_0) \right) \prod_{i=0}^n \left(\int_E u(b_i; a_{i+1}) d(a_{i+1}; b_{i+1}) d\mu(a_{i+1}) \right) \\
&\quad \left(\int_E u(b_{n+1}; a_{n+2}) d\mu(a_{n+2}) \right) \prod_{i=0}^n t(b_i, b_{i+1}; c_i) d\mu^{2n+3}(b_0, \dots, b_{n+1}, c_0, \dots, c_n) \\
&= \int_{F_{2n+1}} r_0(b_0) \prod_{i=0}^n \overline{ud}(b_i; b_{i+1}) t(b_i, b_{i+1}; c_i) d\mu^{2n+3}(b_0, \dots, b_{n+1}, c_0, \dots, c_n) \\
&= \int_{F_{2n+1}} r_0(b_0) \prod_{i=0}^n d(b_i; c_i) u(c_i; b_{i+1}) d\mu^{2n+3}(b_0, \dots, b_{n+1}, c_0, \dots, c_n).
\end{aligned}$$

This shows that the push forward measure of a (ρ_0, D, U) -HZMC is a (ρ_0, D, U) -HZMC. Hence, the (ρ_0, D, U) -HZMC is an invariant measure of **A**.

4.2 Proof of Theorem 1.9

In the case of a μ -positive HZMC, taking \tilde{E} or E does not make any difference in Theorem 1.8. Indeed, by basic properties of measurability: for any property P , $P(x)$ holds for μ -almost $x \in E$ if and only if $P(x)$ holds for μ -almost $x \in \text{supp}(\mu) \cap E$ (set equal to \tilde{E} here). In addition, for a μ -positive (ρ_0, D, U) -HZMC: for μ^2 -almost $(a, b) \in E^2$, $\overline{du}(a, b) > 0$.

To prove Theorem 1.9, we first prove Lemmas 4.1 and 4.2.

Lemma 4.1. *Let **A** be a μ -positive PCA with t.k. T . The three conditions: **Cond 1**, **Cond 4** and*

Cond 9: *For μ^6 -almost (a, a', b, b', c, c') ,*

$$t(a, b; c) t(a, b'; c') t(a', b; c') t(a', b'; c) = t(a', b'; c') t(a', b; c) t(a, b'; c) t(a, b; c'). \quad (10)$$

are equivalent.

Proof. • From **Cond 1** to **Cond 9**: replace in **Cond 9** the expressions of t by the ones given in **Cond 1**.

- From **Cond 9** to **Cond 4**: we prove its contrapositive. Suppose that, for all (a_0, b_0, c_0) , **Cond 4** is false. Hence, for all $(a_0, b_0, c_0) \in E^3$, either $T(a_0, b_0, \cdot)$ and μ are not positive equivalent, or

$$\mu^3(\{(a, b, c) \text{ such that Eq (4) does not hold}\}) > 0. \quad (11)$$

But, by definition of μ -positivity, the set of (a_0, b_0) such that $T(a_0, b_0; \cdot)$ and μ are not positive equivalent is μ^2 -negligible. Hence, for μ^3 -almost (a_0, b_0, c_0) , Eq (11) holds. But, by Fubini theorem,

$$\begin{aligned} & \mu^6(\{(a, b, c, a', b', c') \text{ such that Eq (10) does not hold}\}) \\ &= \int_{E^3} \mu^3(\{(a, b, c) \text{ such that Eq (4) does not hold}\}) d\mu(a_0, b_0, c_0) > 0 \end{aligned}$$

and on the other hand **Cond 9** is equivalent to

$$\mu^6(\{(a, b, c, a', b', c') \text{ such that Eq (10) does not hold}\}) = 0.$$

- From **Cond 4** to **Cond 1**: set

$$d(a; c) = K_a \frac{t(a, b_0; c)}{t(a_0, b_0; c)} \frac{\int_E t(a_0, b; c) d\mu(b)}{\int_E \frac{t(a, b_0; x)}{t(a_0, b_0; x)} d\mu(x)} \text{ and } u(c; b) = \frac{t(a_0, b; c)}{\int_E t(a_0, x; c) d\mu(x)}$$

where K_a is a normalization constant such that $\int_E d(a; c) d\mu(c) = 1$. Then,

$$\overline{du}(a; b) = K_a \int_E \frac{t(a, b_0; c) t(a_0, b; c)}{t(a_0, b_0; c)} \frac{1}{\int_E \frac{t(a, b_0; x)}{t(a_0, b_0; x)} d\mu(x)} d\mu(c),$$

and

$$\frac{d(a; c) u(c; b)}{\overline{du}(a; b)} = \frac{t(a, b_0; c) t(a_0, b; c)}{t(a_0, b_0; c)} \frac{1}{\int_E \frac{t(a, b_0; x) t(a_0, b; x)}{t(a_0, b_0; x)} d\mu(x)} \quad (12)$$

$$= \frac{t(a, b_0; c) t(a_0, b; c)}{t(a_0, b_0; c)} \frac{1}{\int_E t(a, b; x) \frac{t(a, b_0; c_0) t(a_0, b; c_0)}{t(a_0, b_0; c_0) t(a, b; c_0)} d\mu(x)} \quad (13)$$

$$= \frac{t(a, b_0; c) t(a_0, b; c) t(a_0, b_0; c_0) t(a, b; c_0)}{t(a_0, b_0; c) t(a, b_0; c_0) t(a_0, b; c_0)} \frac{1}{\int_E t(a, b; x) d\mu(x)} \quad (14)$$

$$= t(a, b; c). \quad (15)$$

In this previous computation, we pass from line (12) to line (13) and from line (14) to line (15) by using **Cond 4**. \square

Lemma 4.1 says that **Cond 1** is equivalent to **Cond 4** for μ -positive PCA. Next Lemma 4.2, gives some necessary conditions for a (ρ_0, D, U) -HZMC to be invariant by a μ -positive PCA.

Lemma 4.2. *Let \mathbf{A} be a μ -positive PCA. If \mathbf{A} satisfies the conditions of Lemma 4.1, then there exists H a μ -positive probability distribution on $(E, \mathcal{B}(E))$ of μ -density η such that the respective μ -densities of D and U are, for μ^3 -almost (a, b, c) , d^η and u^η as defined in Eq (3).*

Proof. Suppose that, for μ^3 almost (a, b, c) ,

$$\overline{du}(a; b) = \frac{d(a; c)u(c; b)}{t(a, b; c)} = \frac{d(a; c_0)u(c_0; b)}{t(a, b; c_0)}. \quad (16)$$

Then $d(a; c)u(c; b) = d(a; c_0)\frac{u(c_0; b)}{t(a, b; c_0)}t(a, b; c)$. Integrating with respect to b ,

$$d(a; c) = d(a; c_0) \int_E \frac{u(c_0; b)}{t(a, b; c_0)} t(a, b; c) d\mu(b)$$

and, then,

$$u(c; b) = \frac{\frac{u(c_0; b)}{t(a, b; c_0)} t(a, b; c)}{\int_E \frac{u(c_0; x)}{t(a, x; c_0)} t(a, x; c) d\mu(x)}. \quad (17)$$

Cond 4 and **Cond 9** permit to replace a by a_0 on the right side of Eq (17). Then, taking $\eta(b) = u(c_0; b)$ ends the proof. \square

Now, we can end the proof of Theorem 1.9.

Proof of Theorem 1.9. Let \mathbf{A} be a μ -positive PCA. If (ρ_0, D, U) is an invariant HZMC for \mathbf{A} , then there exists $\eta \in L^1(\mu)$ such that Eq (3) holds by **Cond 1**, Lemma 4.1 and Lemma 4.2.

Moreover, u and d satisfy **Cond 2**. Hence, rewriting \overline{du} and \overline{ud} in terms of η ,

$$\overline{du}(a; b) = \int_E \frac{\frac{\eta(x)}{t(a, x; c_0)} t(a, x; c) d\mu(x)}{\int_E \frac{\eta(x)}{t(a, x; c_0)} d\mu(x)} \frac{\frac{\eta(b)}{t(a_0, b; c_0)} t(a_0, b; c)}{\int_E \frac{\eta(x)}{t(a_0, x; c_0)} t(a_0, x; c) d\mu(x)} d\mu(c) \quad (18)$$

$$= \int_E \frac{\frac{\eta(x)}{t(a, x; c_0)} t(a, b; x) d\mu(x)}{\int_E \frac{\eta(x)}{t(a, x; c_0)} d\mu(x)} \frac{\frac{\eta(b)}{t(a, b; c_0)} t(a, b; c)}{\int_E \frac{\eta(x)}{t(a, x; c_0)} t(a, x; c) d\mu(x)} d\mu(c) \quad (19)$$

$$= \frac{\frac{\eta(b)}{t(a, b; c_0)}}{\int_E \frac{\eta(x)}{t(a, x; c_0)} d\mu(x)},$$

we pass from line (18) to line (19) replacing $\frac{t(a_0, b; c)t(a_0, x; c_0)}{t(a_0, b; c_0)t(a_0, x; c)}$ by $\frac{t(a, b; c)t(a, x; c_0)}{t(a, b; c_0)t(a, x; c)}$ using **Cond 4** and **9**; and

$$\overline{ud}(a; b) = \int_E \frac{\frac{\eta(c)}{t(a_0, c; c_0)} t(a_0, c; a)}{\int_E \frac{\eta(x)}{t(a_0, x; c_0)} t(a_0, x; a) d\mu(x)} \frac{\int_E \frac{\eta(x)}{t(c, x; c_0)} t(c, x; b) d\mu(x)}{\int_E \frac{\eta(x)}{t(c, x; c_0)} d\mu(x)} d\mu(c).$$

Hence, η satisfies Eq (4) of **Cond 5**.

Finally, we need a distribution ρ_0 to satisfy **Cond 3** with $D = D^\eta$, this is possible only if **Cond 6** holds.

Conversely, if we suppose **Cond 4**, **Cond 5** and **Cond 6**, then all the previous computations hold and then we obtain **Cond 1**, **Cond 2** and **Cond 3** for $D = D^\eta$, $U = U^\eta$ and ρ_0 and then we conclude using Theorem 1.8. \square

4.3 Proof of Proposition 1.11

Let **A** be a PCA and suppose that **Cond 4**, **Cond 7** and **Cond 8** hold. Then, we can replace in **Cond 5** the a_0 by c using **Cond 4** and **Cond 7**. Then η must verify: for μ -almost a and for the c_0 of **Cond 4**,

$$\frac{\frac{\eta(a)}{t(a, a; c_0)}}{\int_E \frac{\eta(x)}{t(a, x; c_0)} d\mu(x)} = \int_E \frac{\frac{\eta(c)}{t(c, c; c_0)}}{\int_E \frac{\eta(x)}{t(c, x; c_0)} d\mu(x)} t(c, c; a) d\mu(c).$$

So, we see that $\left(a \mapsto \frac{\frac{\eta(a)}{t(a, a; c_0)}}{\int_E \frac{\eta(x)}{t(a, x; c_0)} d\mu(x)} \right)$ is an eigenfunction of the operator $\mathcal{A}_1 : f \mapsto \left(\mathcal{A}_1(f) : a \mapsto \int_E f(c) t(c, c; a) d\mu(c) \right)$. Hence, by Lemma 1.13, if there exists a positive eigenfunction ν in $L^1(\mu)$ for \mathcal{A}_1 , it is unique up to a multiplicative constant. Hence, there exists $\lambda > 0$ such that, for μ -almost a ,

$$\frac{\frac{\eta(a)}{t(a, a; c_0)}}{\int_E \frac{\eta(x)}{t(a, x; c_0)} d\mu(x)} = \lambda \nu(a),$$

which is equivalent to

$$\eta(a) = \lambda \int_E \eta(x) \frac{t(a, a; c_0)}{t(a, x; c_0)} \nu(a) d\mu(x).$$

Hence, η is an eigenfunction of $\mathcal{A}_2 : f \mapsto \left(\mathcal{A}_2(f) : a \mapsto \int_E f(x) \frac{t(a, a; c_0)}{t(a, x; c_0)} \nu(a) d\mu(x) \right)$.

5 Extension to \mathbb{Z} and $\mathbb{Z}/n\mathbb{Z}$

5.1 PCA on \mathbb{Z}

In this section, we extend Theorem 1.8 and 1.9 to \mathbb{Z} . The main change is that ρ_0 the initial probability distribution for a HZMC on \mathbb{N} is replaced on \mathbb{Z} by a sequence of probability distributions $R = (\rho_i)_{i \in \mathbb{Z}}$ indexed by \mathbb{Z} .

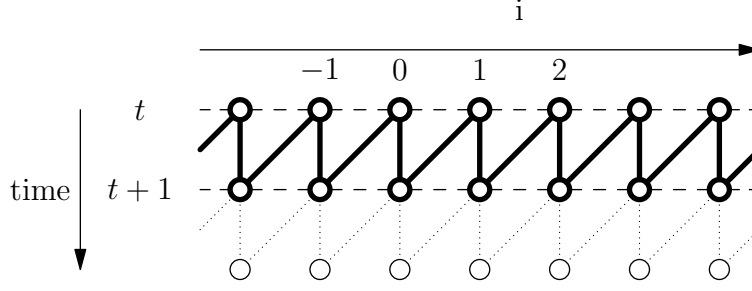


Figure 2: In bold, $\text{HZ}_{\mathbb{Z}}(t)$, the t th horizontal zigzag on \mathbb{Z} on a space time diagram.

Let us define a $\text{HZMC}_{\mathbb{Z}}$ on \mathbb{Z} . The geometrical structure is now

$$\text{HZ}_{\mathbb{Z}}(t) = \left\{ \left(\left\lfloor \frac{i}{2} \right\rfloor, t + \frac{1 + (-1)^{i+1}}{2} \right), i \in \mathbb{Z} \right\}.$$

See Figure 2 for a graphical representation.

On this structure, a (R, D, U) - $\text{HZMC}_{\mathbb{Z}}$ is a Markov chain with two t.k. D and U and a family of probability distributions $R = (\rho_i)_{i \in \mathbb{Z}}$ such that

- for all $i \in \mathbb{Z}$, the distribution of state $S(i, t)$ is ρ_i ,
- the distribution of $S(i, t+1)$ knowing $S(i, t)$ is given by D , and that of $S(i+1, t)$ knowing $S(i, t+1)$ is given by U .

Hence, for any $i \in \mathbb{Z}$, the distributions ρ_i , ρ_{i+1} , D and U are constrained such that $\rho_i D U = \rho_{i+1}$. In the case of a μ -supported (R, D, U) - $\text{HZMC}_{\mathbb{Z}}$ (i.e. for all $i \in \mathbb{Z}$, $\rho_i \ll \mu$ and D and U are μ -supported t.k.), this gives

$$r_{i+1}(x_{i+1}) = \int_E r_i(x_i) \overline{du}(x_i; x_{i+1}) d\mu(x_i). \quad (20)$$

A family of probability distributions R that possesses this property is said to be compatible with (D, U) .

As before, we define the support $\tilde{E} = \bigcup_{i \in \mathbb{Z}} \text{supp}(\rho_i)$ of an $\text{HZMC}_{\mathbb{Z}}$. If the $\text{HZMC}_{\mathbb{Z}}$ is μ -supported, then, for μ -almost $a \in E$, there exists i such that $r_i(a) > 0$ and, in the case of a μ -positive $\text{HZMC}_{\mathbb{Z}}$ (i.e. for all $i \in \mathbb{Z}$, ρ_i and μ are positive equivalent and D and U are μ -positive t.k.), $\tilde{E} = \text{supp}(\mu)$.

The following theorem is an extension of Theorem 1.8 for PCA on \mathbb{Z} .

Theorem 5.1. *Let μ be a σ -finite measure on E . Let $\mathbf{A} := (\mathbb{Z}, E, N, T)$ be a μ -supported PCA and (R, D, U) a μ -supported $\text{HZMC}_{\mathbb{Z}}$. The (R, D, U) - $\text{HZMC}_{\mathbb{Z}}$ is invariant by \mathbf{A} if and only if **Cond 1**, **Cond 2** and the following condition are satisfied:*

Cond 10: ρ is invariant by t.k. D , i.e. for all $i \in \mathbb{Z}$, $\rho_i D = \rho_i$, i.e. for all $i \in \mathbb{Z}$, $r_i(c) = \int_E r_i(a) d(a; c) d\mu(a)$.

Proof. This is an immediate consequence of Theorem 1.8 because we just need, for all $i \in \mathbb{Z}$, the (ρ_i, D, U) -HZMC to be invariant by \mathbf{A} . \square

As in Theorem 1.9 where we go further for μ -positive PCA on \mathbb{N} , we obtain a necessary and sufficient condition on μ -positive PCA to have an invariant HZMC $_{\mathbb{Z}}$.

Theorem 5.2. *Let μ be a σ -finite measure on E . Let $\mathbf{A} := (\mathbb{Z}, E, N, T)$ be a μ -positive PCA. \mathbf{A} admits a μ -positive invariant HZMC $_{\mathbb{Z}}$ if and only if **Cond 4**, **Cond 5** and **Cond 6** hold.*

In this case, the (R, D, U) -HZMC $_{\mathbb{Z}}$ has for μ -densities d^n and u^n as defined in Eq (3) and, for any $i \in \mathbb{Z}$, $\rho_i = \rho_0$.

Proof. It is an immediate consequence of Theorem 1.9. The only new thing to verify is that $R = \rho_0^{\mathbb{Z}}$ is (D, U) compatible, i.e. r_0 satisfies Eq (20) to check that (R, D, U) defines a HZMC $_{\mathbb{Z}}$. By **Cond 6**, for μ -almost y_i ,

$$\int_E r_0(x_i) d(x_i; y_i) d\mu(x_i) = r_0(y_i). \quad (21)$$

But, satisfying Eq (20) and Eq (21) is equivalent to satisfy, for μ -almost x_{i+1} ,

$$\int_E r_0(y_i) u(y_i; x_{i+1}) d\mu(y_i) = r_0(x_{i+1}).$$

Now, from Eq (21), for μ -almost x_{i+1} ,

$$\int \int_{E^2} r_0(x_i) d(x_i; y_i) u(y_i; x_{i+1}) d\mu(x_i) d\mu(y_i) = \int_E r_0(y_i) u(y_i; x_{i+1}) d\mu(y_i).$$

But as $\overline{du} = \overline{ud}$,

$$\int_E \left(\int_E r_0(x_i) u(x_i; y_i) d\mu(x_i) \right) d(y_i; x_{i+1}) d\mu(y_i) = \int_E r_0(y_i) u(y_i; x_{i+1}) d\mu(y_i).$$

So, $f : y \rightarrow \int_E r_0(x) u(x; y) d\mu(x)$ is a positive eigenfunction of the integral operator \mathcal{A} of kernel d . By Lemma 1.13, this eigenfunction is unique (up to a multiplicative constant) equal to r_0 , so $\int_E r_0(x) u(x; y) d\mu(x) = \lambda r_0(y)$ and $\lambda = 1$ because they both integrates (with respect to μ) to 1. This ends the proof. \square

In that case, Prop 1.11 still holds and Prop 2.4 also holds if (ρ_0, D, U) -HZMC is replaced by (R, D, U) -HZMC $_{\mathbb{Z}}$.

5.2 PCA on $\mathbb{Z}/n\mathbb{Z}$

In this section, we have results, similar to Theorems 1.8 and 1.9, on the lattice $\mathbb{Z}/n\mathbb{Z}$. The main change is that we characterize PCA whose invariant distribution is a cyclic-HZMC (CHZMC).

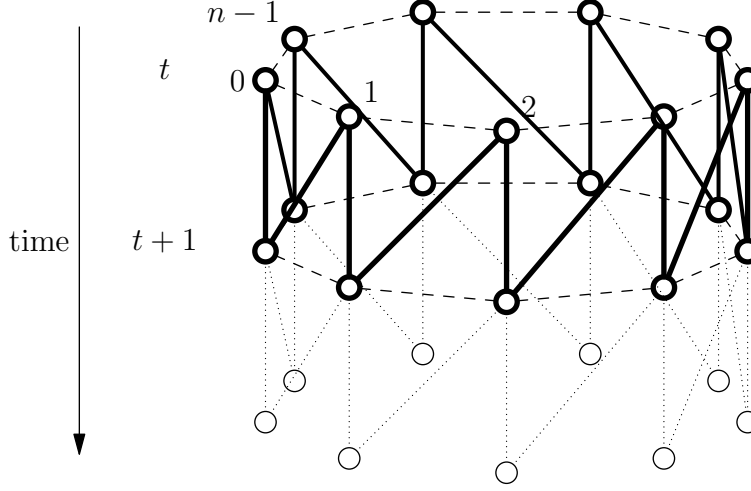


Figure 3: In bold, $\text{CHZ}(t)$, the t th cyclic horizontal zigzag on a space time diagram.

Consider, as represented on Figure 3,

$$\text{CHZ}(t) = \left\{ \left(\left\lfloor \frac{i}{2} \right\rfloor, t + \frac{1 + (-1)^{i+1}}{2} \right), i \in \mathbb{Z}/(2n)\mathbb{Z} \right\}.$$

Let (D, U) be two μ -supported t.k. from E to E such that

$$Z(D, U) = \int_{E^{2n}} u(y_{n-1}; x_0) d(x_0; y_0) \dots d(x_{n-1}; y_{n-1}) d\mu^{2n}(x_0, y_0, x_1, \dots, y_{n-1}) \notin \{0, +\infty\}.$$

We define the measure M on CHZ called $(\mu$ -supported) (D, U) -CHZMC by its μ^{2n} -density m that is: for μ -almost $x_0, y_0, \dots, y_{n-1} \in E$,

$$m(x_0, y_0, \dots, y_{n-1}) = \frac{u(y_{n-1}; x_0) d(x_0; y_0) \dots d(x_{n-1}; y_{n-1})}{Z(D, U)}.$$

For simplicity, we define, formally, only μ -supported (D, U) -CHZMC (D and U are μ -supported t.k. from E to E).

When E is finite, a CHZMC is a HZMC conditioned to be periodic. In general, a CHZMC is a Gibbs measure in the cyclic horizontal zigzag (CHZ).

Cyclic Markov chain were introduced, first, by Albenque [1] to define periodic Markov chain on $\mathbb{Z}/n\mathbb{Z}$.

Notation. The distribution of the line S_t (resp. S_{t+1}) is denoted $M^{(1)}$ (resp. $M^{(2)}$) and its μ^n -density is obtained by integration of m with respect to the n variables y_0, \dots, y_{n-1} (resp. to the n variables x_0, \dots, x_{n-1}). The distribution of the state $S(i, t)$ is denoted $M_i^{(1)}$ and its μ -density is obtained by integration of m with respect to the $2n - 1$ variables $x_0, y_0, \dots, x_{i-1}, y_{i-1}, y_i, x_{i+1}, \dots, x_{n-1}$.

For any $j \in \mathbb{N}$, for μ -almost a, b , we let

$$\overline{(du)^j}(a; b) = \int_{E^{2j-1}} d(a; y_0) u(y_0; x_1) \dots u(y_{j-1}; b) d\mu^{2j-1}(y_0, x_1, \dots, y_{j-1}).$$

We obtain, first, a theorem about μ -supported PCA having μ -supported CHZMC.

Theorem 5.3. *Let μ be a σ -finite measure on E . Let $\mathbf{A} := (\mathbb{Z}/n\mathbb{Z}, E, N, T)$ be a μ -supported PCA and (D, U) a μ -supported CHZMC. The (D, U) -CHZMC is invariant by \mathbf{A} if and only if the two following conditions are satisfied:*

Cond 11: for μ -almost $a, b, c \in E$,

$$\overline{du}(a; b)t(a, b; c) = d(a; c)u(c; b) \text{ or } \overline{(du)^{n-1}}(b; a) = 0,$$

Cond 12: for μ -almost $x_0, x_1, \dots, x_{n-1} \in \tilde{E}$,

$$\overline{du}(x_0; x_1)\overline{du}(x_1; x_2) \dots \overline{du}(x_{n-1}; x_0) = \overline{ud}(x_0; x_1)\overline{ud}(x_1; x_2) \dots \overline{ud}(x_{n-1}; x_0)$$

Proof. Let a (D, U) -CHZMC be invariant by \mathbf{A} . For all $A, B, C \in \mathcal{B}(E)$, for all $i \in \mathbb{Z}/n\mathbb{Z}$,

$$\begin{aligned} & P(S(i, t) \in A, S(i+1, t) \in B, S(i, t+1) \in C) \\ &= \frac{1}{Z(D, U)} \int_{A \times C \times B} d(x_i; y_i)u(y_i; x_{i+1})\overline{(du)^{n-1}}(x_{i+1}; x_i)d\mu^3(x_i, y_i, x_{i+1}) \\ &= \frac{1}{Z(D, U)} \int_{A \times C \times B} \overline{du}(x_i; x_{i+1})t(x_i, x_{i+1}; y_i)\overline{(du)^{n-1}}(x_{i+1}; x_i)d\mu^3(x_i, y_i, x_{i+1}). \end{aligned}$$

Hence, for μ -almost $x_i, y_i, x_{i+1} \in E$,

$$\overline{du}(x_i; x_{i+1})t(x_i, x_{i+1}; y_i)\overline{(du)^{n-1}}(x_{i+1}; x_i) = d(x_i; y_i)u(y_i; x_{i+1})\overline{(du)^{n-1}}(x_{i+1}; x_i),$$

that is **Cond 11**.

To prove **Cond 12**, we use the fact that the second line of the (D, U) -CHZMC at time t is the first line at time $t+1$ and since the CHZMC is invariant the law of the CHZMC at time t and at time $t+1$ is the same M . But $M^{(1)}$ is the law of the first line and $M^{(2)}$ of the second, so $M^{(1)} = M^{(2)}$. In terms of μ^n -densities, $m^{(1)} = m^{(2)}$. But, $m^{(1)}(x_0, \dots, x_{n-1}) = \frac{1}{Z(D, U)}\overline{du}(x_0; x_1) \dots \overline{du}(x_{n-1}; x_0)$ and $m^{(2)}(x_0, \dots, x_{n-1}) = \frac{1}{Z(D, U)}\overline{ud}(x_0; x_1) \dots \overline{ud}(x_{n-1}; x_0)$ that gives **Cond 12**.

Conversely, we suppose that **Cond 11** and **Cond 12** are satisfied. Then, the push forward measure of the (D, U) -CHZMC by \mathbf{A} is also the (D, U) -CHZMC (the computation is an adaptation of that done in the Proof of Theorem 1.8 in Section 4.1 to compute the push forward measure of a HZMC). \square

For μ -positive PCA, **Cond 11** could be exploited a little more.

Theorem 5.4. *Let μ be a σ -finite measure on E . Let $\mathbf{A} := (\mathbb{Z}/n\mathbb{Z}, E, N, T)$ be a μ -positive PCA. \mathbf{A} admits a μ -positive invariant CHZMC if and only if **Cond 4** and*

Cond 13: *there exists a positive function $\eta \in L^1(\mu)$ solution of: for μ -almost $x_0, \dots, x_{n-1} \in E$,*

$$\overline{d^\eta u^\eta}(x_0; x_1)\overline{d^\eta u^\eta}(x_1; x_2) \dots \overline{d^\eta u^\eta}(x_{n-1}; x_0) = \overline{u^\eta d^\eta}(x_0; x_1)\overline{u^\eta d^\eta}(x_1; x_2) \dots \overline{u^\eta d^\eta}(x_{n-1}; x_0)$$

with d^n and u^n as defined in Eq (3).

In this case, the (D, U) -CHZMC has for μ -densities d^n and u^n as defined in Eq (3).

Proof. First of all, when a PCA is μ -positive, **Cond** 11 can be rewritten, for μ -almost a, b, c , $t(a, b; c) = \frac{d(a; c)u(c; b)}{\overline{du}(a; b)}$ because both $\overline{du}(a; b)$ and $\overline{(du)^{n-1}}(b; a)$ are positive. Hence, we use Lemma 4.1 to prove that **Cond** 11 is equivalent to **Cond** 4. Moreover, Lemma 4.2 still applies and the state space of possible solutions for (D, U) is parametrized by η a function in $L^1(\mu)$. **Cond** 12 applied on d^n and u^n gives **Cond** 13. \square

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